Subject: الاحتمالات

بسم الله الرحمن الرحيم

① Probability

② Random variables

③ Random processes

ch 1: probability.

1- set Definitions: Group of elements

$A = \{a, b, c, d, e, f\}$

$\hookrightarrow$ set A

$\hookrightarrow$ set A contains the following elements

$a \in A$

element a belongs to set A

$b \in A$

$h \notin A$: element h does not belong to set A

Can we comt the number of elements in a given set?

1) if the number of elements is couted

$\Rightarrow$ contable set $|A| = 6$

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2) if the number of elements is not contad

→ uncountable set

* set of real number between 1 and 2

1.0002 , 1.000002 , 1.000001 , 1.0001

uncountable set

EX/ set B of all integer numbers between 1 and 9, inclusive? شاملة

$B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $|B| = 9$

EX/ set c of all integer numbers between 1 and 9, exclusive? حصرية

$c = \{2, 3, 4, 5, 6, 7, 8\}$ $|c| = 7$

$|\,|$ → rank = number of elements in a set

تقاطع

$\underbrace{B \cap C}_{\text{intersection}} = \{2, 3, 4, 5, 6, 7, 8\}$

intersection التقاطع

the common elements between set B and C

$A \cap B = \{\,\} = \emptyset$ → empty set

$|\{\,\}| = $ zero

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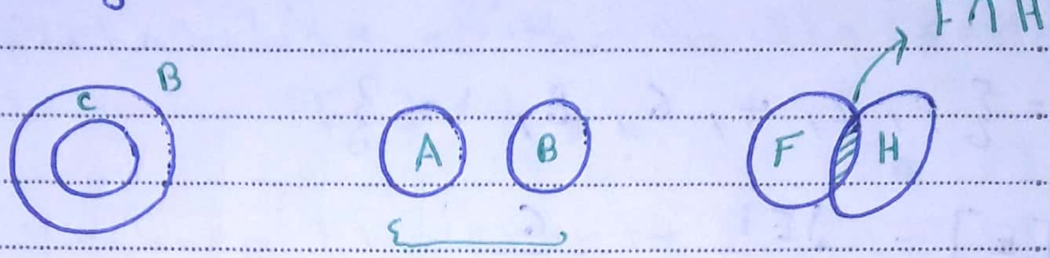
in the case that $A \cap B = \phi$ ⟹ that we have
- disjoint events ← $[A \cap B = \phi]$
- mutually exclusive events ←

Venn Diagram



disjoint Events

$\underbrace{B \cup C}_{}$ (union) $= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

↳ the element should be a vailable at least either B or C

$C \cup A = \{a, b, c, d, e, f, 2, 3, 4, 5, 6, 7, 8\}$

sample space [s] : set of all possible out comes

Experiment : set of all integers between 0 and 10, inclusive ?

$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

what is the sample space ?

1) set of odd numbers

$O = \{1, 3, 5, 7, 9\}$

$P[O] = \dfrac{|O|}{|S|} = \dfrac{5}{11}$

2) set of even numbers

$E = \{0, 2, 4, 6, 8, 10\}$

$P[E] = \dfrac{|E|}{|S|} = \dfrac{6}{11}$

$P[S] = \dfrac{|S|}{|S|} = 1$

$P[\{\}] = \dfrac{|\{\}|}{|S|} = \dfrac{0}{11} = \text{zero}$

قطعة نقدية

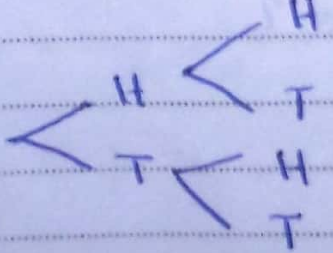
Ex/ Tossing a fair coin and noticing the result?

$S = \{H, T\}$

since we have a fair coin

$P[H] = \dfrac{1}{2}$ $P[T] = \dfrac{1}{2}$

Experinant : Tossing a fair coin twice and Absrring the out comes



$S = \{HH, HT, TH, TT\}$

Axioms of Probability

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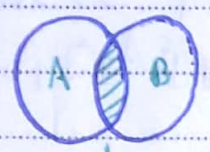
1) The probability of any event is between 0 and 1

$P[\{\}] \leftarrow 0 \le P[A] \le 1 \rightarrow P[S]$

2) The probability of the sample space [S] is $P[S] = 1$

Ex: Let we have two events A and B as shown in the Venn Diagram

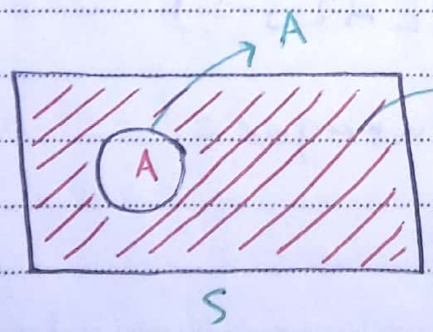


what is $P[A \cup B]$?

$\hookrightarrow A \cap B$

$A \cup B = A + B - A \cap B$

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$



$\bar{A} = S - A$

$\bar{A}$: A complimant : all elements that are in S but not in A

$A \cap \bar{A} = \{\}, \phi$

$A \cup \bar{A} = S$

$\hookrightarrow P[A \cap \bar{A}] = 0$

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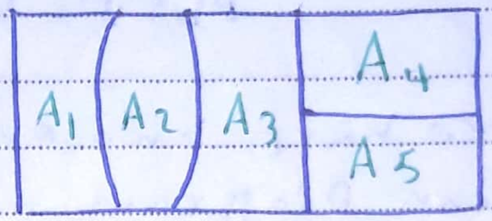
$$P[A \cup \bar{A}] = P[A] + P[\bar{A}] - \cancelto{0}{P[A \cap \bar{A}]}$$

$$= P[A] + P[\bar{A}] = P[S] = 1$$

Axiom # 3

Let S is divided into non_intersected events



$A_1 \cap A_2 = \{\}$

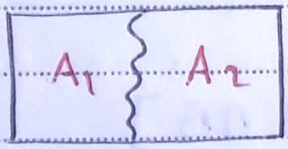
$A_1 \cap A_3 = \{\}$

$\vdots$

$P[A_1 \cup A_2 \cup A_3 \cup A_4 \dots A_n]$

$= P[A_1] + P[A_2] + P[A_3] + \dots + P[n]$

$= 1$



$A_1 \cap A_2 = \phi \quad A_1 \cup A_2 = S$

$P[A_1 \cup A_2] = P[A_1] + P[A_2] -$

$\cancelto{0}{P[A_1 \cap A_2]} = P[A_1] + P[A_2] = 1$

EX / Let S is given by all integers

$S = \{1 \le \text{integers} \le 12\}$

$= \{1, 2, 3, 4, \dots, 12\}$

Let we have the following events

$A = \{1, 3, 5, 12\}$

$B = \{2, 6, 7, 8, 9, 10, 11\}$

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$C = \{ 1, 3, 4, 6, 7, 8 \}$

$\bar{A} = S - A$

$= \{ 2, 4, 6, 7, 8, 9, 10, 11 \}$

$\bar{B} = S - B$

$= \{ \cancel{2}, 4, 3, 5, 12 \}$

$\bar{C} = S - C$

$A \cap B = \phi \rightarrow$ A and B are disjoint events

$\Rightarrow P[A \cap B] = 0$

$A \cup B = \{ 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12 \}$

$P[A \cup B] = \dfrac{|A \cup B|}{|S|} = \dfrac{11}{12}$

$P[A] = \dfrac{|A|}{|S|} = \dfrac{4}{12}$

$P[\bar{A}] = \dfrac{8}{12}$

$P[A] + P[\bar{A}] = 1 \Rightarrow \dfrac{4}{12} + \dfrac{8}{12} = 1$

Algebra of sets

1) commutative Law

$A \cap B = B \cap A \qquad P[A \cap B] = P[B \cap A]$

$A \cup B = B \cup A \qquad P[A \cup B] = P[B \cup A]$

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2) Distributive Law

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3) Associative Law

$A \cup (B \cup C) = B \cup (A \cup C) = C \cup (B \cup A)$
$A \cap (B \cap C) = B \cap (A \cap C) = C \cap (B \cap A)$

De - Morgan's Law

$\bar{A} = S - A$
$\overline{(A \cap B)} = \bar{A} \,\bar{\cap}\, \bar{B} = \bar{A} \cup \bar{B}$

$\bar{\cap} = \cup$
$\bar{\cup} = \cap$

$\overline{A \cap B} = S - (A \cap B)$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\bar{\bar{A}} = A$

Ex / $\overline{A \cap (B \cup C)} = \bar{A} \cup (\overline{B \cup C})$
$= \bar{A} \cup (\bar{B} \cap \bar{C})$

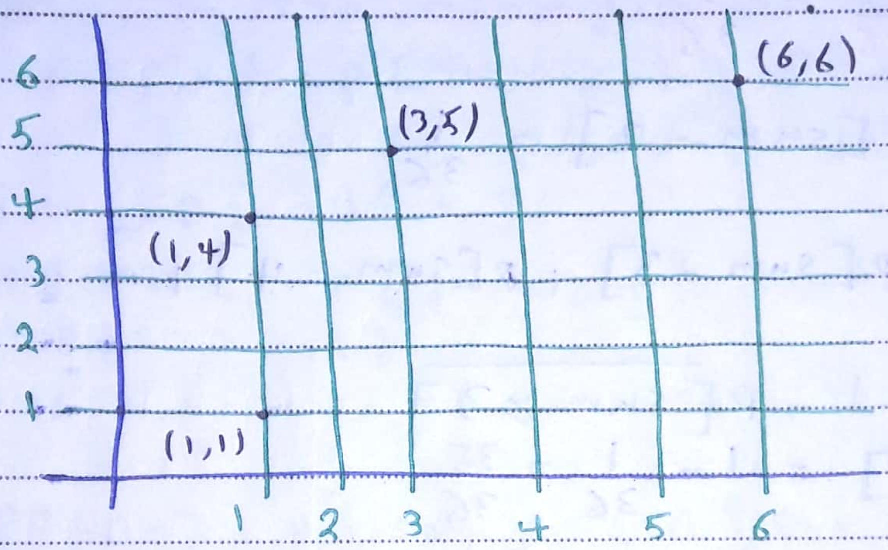
$\overline{\overline{B \cap C}} = B \cap C$
$\overline{\bar{B} \cap \bar{C}} = \bar{\bar{B}} \cup \bar{\bar{C}} = B \cup C$

Ex / An experiment consist of observing the sum of the numbers showing up when two dice are thrown

1) what is the sample space ?

$S = \{ 2 \leq x \leq 12 ,\ x \text{ is integer} \}$

or 2, 3, 4, 5, 6, ..., 12



Number of out comes = 36

$P[\text{sum} = 2] = \frac{1}{36} = \frac{|(1,1)|}{|S|}$

$\searrow \{(1,1)\}$

$|S| = 36$

$\{ (1,1) , (1,2) , (1,3) , (1,4) , (1,5) , (1,6)$

(2,1) // // // // //

(3,1) // // // // //

// // // // // //

// // // // // //

// // // // // $(6,6) \}$

$P[\text{sum} = 3] = \frac{|\{(1,2),(2,1)\}|}{|S|} = \frac{2}{36} = \frac{1}{18}$

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$P[sum = 7] = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$P[sum = 7] = \frac{6}{36} = \frac{1}{6}$

$P[sum \leq 3] = P[sum = 2] + P[sum = 3]$

$= \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$

$P[sum < 3] = P[sum = 2] = \frac{1}{36}$

$P[sum \geq 3] = P[sum = 3] + P[sum = 4] + \ldots\ldots + P[sum = 12]$

$= 1 - P[\overline{sum \geq 3}]$

$= 1 - P[sum < 3] = 1 - \frac{1}{36} = \frac{35}{36}$

$P[sum \geq 3] + P[\overline{sum \geq 3}] = 1$

$P[sum > 3] = P[sum = 4] + \ldots\ldots + P[sum = 12]$

$= 1 - P[\overline{sum > 3}] = 1 - P[sum \leq 3]$

$1 - P[sum = 2] - P[sum = 3]$

$= 1 - \frac{3}{36} = \frac{33}{36}$

Joint probability

$A \cap B = \phi = \{\}$ $\quad P[A \cap B] = 0$

- disjoint events
- mutually exclusive

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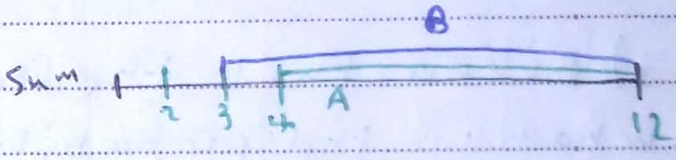
Joint probability

$P[A \cup B] = P[A] + P[B] - P[\overset{\text{joint}}{A \cap B}]$

$\Rightarrow P[A \cap B] = P[A] + P[B] - P[A \cup B]$

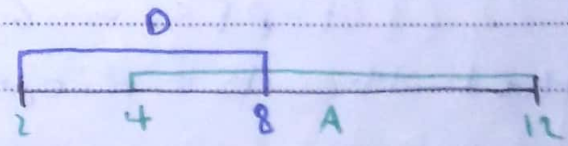
Ex/ In the example of two dice Let the event $A = P[\text{sum} \geq 4]$, $D = P[\text{sum} \leq 8]$
event $B = P[\text{sum} > 3]$, $C = P[\text{sum} < 3]$
find $P[A \cap B]$?

$P[A \cap B] = P[\{\text{sum} \geq 4\} \cap \{\text{sum} \geq 3\}]$
$= P[\text{sum} \geq 4]$
$= \text{.............}$



$P[A \cap C] = P[\{\text{sum} \geq 4\} \cap \{\text{sum} < 3\}] = 0$

$P[A \cap D] = P[\{\text{sum} \geq 4\} \cap \{\text{sum} \leq 8\}]$
$= P[\{\text{sum} \geq 4\} \leq \text{sum} \leq \{\text{sum} \leq 8\}] = P[4 \leq \text{sum} \leq 8]$
$= P[\text{sum} = 4] + P[\text{sum} = 5] + P[\text{sum} = 6] + P[\text{sum} = 7]$
$+ P[\text{sum} = 8]$



Conditional probability

إذا صار

$P[A/B]$ احتمال يعتمد على حدث قبله

Given that the event B has occuered, what is the probability of event A?

$P[A/B] = \dfrac{P[A \cap B]}{P[B]}$ $\quad P[B] \neq 0$

probability has occuerd happend

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EX/ An urn has 10 white balls and 5 red balls

1) what is the probability that a white ball is selected?

$P[w_1] = \frac{10}{15} = \frac{2}{3}$

2) Given that the 1st selected ball was white, what is the probability that the 2nd ball is white?

$P[w_2 / w_1] = \frac{9}{14}$

3) Given that the 1st selected ball was white what is the probability that the 2nd ball is red?

$P[red / white] = \frac{5}{14}$

EX/ we return back to the example of two dice let event
$A = P[sum = 6]$ what is the probability that the 1st dice is odd?

Let event B: the probability that the first dice is odd.

(1, 5), (2, 4), (3, 3)
(4, 2), (5, 1)

$A = P[sum = 6] = \frac{5}{36}$

$P[B / A] = \frac{P[A \cap B]}{P[A]} = \frac{\frac{3}{36}}{\frac{5}{36}} = \frac{3}{5}$

what is the probability that the 2nd dice is even ~~*~~?

C: the probability that the 2nd dice is even

$$P[C/A] = \frac{P[C \cap A]}{P[A]} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

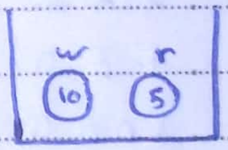
* Joint probability $P[A \cap B]$

* conditional probability

$$P[A/B] = \frac{P[A \cap B]}{P[B]}$$

(3) Given that two balls are selected, what is the probility that the two balls is white?

$$P[A/B] = \frac{P[A \cap B]}{P[B]}$$



$$P[A \cap B] = P[A/B]\, P[B]$$

B: the event that the 1st ball is white

A: the event that the 2nd ball is white

$$P[w_1 \cap w_2] = P[w_2/w_1]\, P[w_1]$$
$$= \left(\frac{9}{14}\right) * \left(\frac{10}{15}\right) = \frac{90}{210}$$

(5) ~~what is the~~ Given that two balls are selected what is the probability that the 2nd ball is white?

$$P[w_2] = P[w_2 \cap r] + P[w_2 \cap w_1]$$
$$= P[w_2/r]\, P[r] + \underbrace{P[w_2/w_1]\, P[w_1]}$$
$$= \left(\frac{10}{14}\right)\left(\frac{5}{15}\right) + \frac{90}{210}$$

$$\frac{50}{210} + \frac{90}{210} = \frac{140}{210}$$

EX: we have 100 resistors in a box as shown in the table: Let a resistor be selected from the box and each resistor has the same likelihood of being chosen Define the following events:

event A: draw a $47\,\Omega$ resistor
event B: draw a resistor whith 5% tolerace
event C: draw a $100\,\Omega$ resistor

resistance	Tolerance 5%	10%	Total
$22\,\Omega$	10	14	24
$47\,\Omega$	28	16	44
$100\,\Omega$	24	8	32
	62	38	100

1) $P[A] = \frac{44}{100}$

2) $P[B] = \frac{62}{100}$

/ [illegible]

3) Given that the selected resister have a 5% toleronce, what is the Probability that the resistor have a $47\,\Omega$ value?

$$P[A/B] = \frac{P[A \cap B]}{P[B]} = \frac{\frac{28}{100}}{\frac{62}{100}} = \frac{28}{62}$$

4) $P[C] = \frac{32}{100}$

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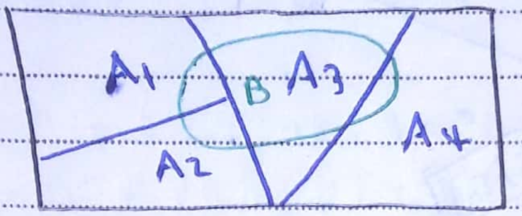
(5) $P[A \cap C] = 0$ disjoint Events [mutally exclusive]

(6) $P[B \cap C] = \frac{24}{100}$

(7) $P[C/B] = \frac{P[B \cap C]}{P[B]} = \frac{\frac{24}{100}}{\frac{62}{100}} = \frac{24}{62}$

Total probability



S

A_1, A_2, A_3, A_4 : disjoint Events

$A_1 \cap A_2 = \phi$ | $A_2 \cap A_3 = \phi$

$A_1 \cap A_3 = \phi$ | $A_2 \cap A_4 = \phi$

$A_1 \cap A_4 = \phi$ | $A_3 \cap A_4 = \phi$

$A_1 \cup A_2 \cup A_3 \cup A_4 = S$

$P[A_1] + P[A_2] + P[A_3] + P[A_4]$

Let we have anoter event B

$B = B \cap S$

$= B \cap [A_1 \cup A_2 \cup A_3 \cup A_4]$

$B = [B \cap A_1] \cup [B \cap A_2] \cup [B \cap A_3] \cup [B \cap A_4]$

$P[B] = P[B \cap A_1] + P[B \cap A_2] + P[B \cap A_3] + P[B \cap A_4]$

↳ total probability of the event B

$P[B/A_1] = \frac{P[A_1 \cap B]}{P[A_1]}$

$P[A_1 \cap B] = P[B/A_1]\ P[A_1]$

$= P[B/A_1]\ P[A_1] + P[B/A_2]\ P[A_2] + P[B/A_3]\ P[A_3]$
$+ P[B/A_4]\ P[A_4]$

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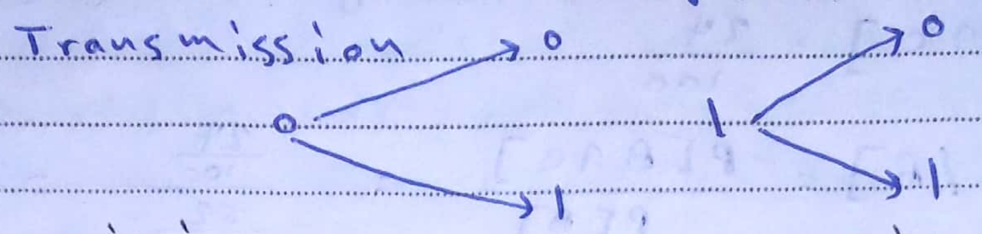
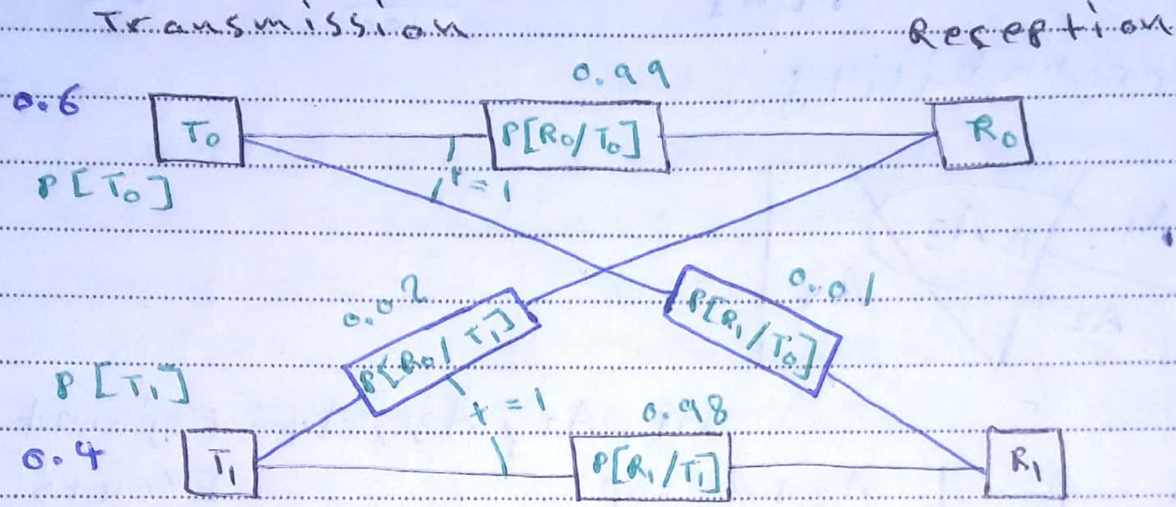
Binary communication channel





$$P[T_0] + P[T_1] = 1$$

(1)

$$P[R_0] = P[T_0 \cap R_0] + P[T_1 \cap R_0]$$

$$= P[R_0/T_0]\,P[T_0] + P[R_0/T_1]\,P[T_1]$$

$$= (0.99)(0.6) + (0.02)(0.4) = 0.594 + 0.008 = 0.602$$

(2) $P[R_1] = P[T_1 \cap R_1] + P[T_0 \cap R_1]$

$$= P[R_1/T_1]\,P[T_1] + P[R_1/T_0]\,P[T_0]$$

$$= (0.98)(0.4) + (0.01)(0.6) = 0.392 + 0.006 = 0.398$$

(3) Given that 0 has been receivel what is the probability that 1 was transmittel?

$$P[T_1/R_0] = \frac{P[T_1 \cap R_0]}{P[R_0]} = \frac{P[R_0/T_1]\,P[T_1]}{P[R_0]}$$

$$= \frac{0.02 * 0.4}{0.602} = \frac{0.008}{0.602}$$

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④ Given that 1 has been received, what is the probability that 1 was Trasmittel?

$$P[T_1 / R_1] = \frac{P[T_1 \cap R_1]}{P[R_1]} = \frac{P[R_1 / T_1]\, P[T_1]}{P[R_1]}$$

$$= \frac{(0.98)(0.4)}{0.398} = \frac{0.392}{0.398}$$

⑤ what is the probability of Error?

$$P[E] = P[T_0 \cap R_1] + P[T_1 \cap R_0]$$

$$= P[R_1 / T_0]\, P[T_0] + P[R_0 / T_1]\, P[T_1]$$

$$= (0.01)(0.6) + (0.02)(0.4) = 0.006 + 0.008 = 0.014$$